

# JEE (Advanced) - 2018 PAPER-1 20-05-2018

# Time : 3 Hours

# Maximum Marks : 180

- This question paper has three (03) parts: PART-I: Physics, PART-II: Chemistry and PART-III: Mathematics.
- Each part has total of eighteen (18) questions divided into three (03) sections (Section-1, Section-2 and Section-3).
- Total number of questions in Paper-1 : Fifty four (54).
- Paper-1 Maximum Marks : One Hundred Eighty (180).

### Instructions for Section-1 : Questions and Marking Scheme

### SECTION-1 (Maximum Marks : 24)

- This section contains **SIX (06)** questions.
- Each question has **FOUR options** for correct answer(s). **ONE OR MORE THAN ONE** of these four option(s) is (are) correct option(s).
- For each question, choose the correct option(s) to answer the question.
- Answer to each question will be evaluated according to the following marking chosen.
  - Full Marks : +4 If only (all) the correct option(s) is (are) chosen.
  - Partial Marks : +3 If all the four options are correct but ONLY three options are chosen.
  - Partial Marks : +2 If three or more options are correct but ONLY two options are chosen, both of which are correct options.
  - Partial Marks : +1 If two or more options are correct but ONLY one option is chosen and it is a correct option.
  - Zero Marks : **0** If none of the options is chosen (i.e. the question is unanswered).
  - Negative Marks : -2 In all other cases.
- For Example : If first, third and fourth are the ONLY three correct options for a question with second option being an incorrect option; selecting only two of the three correct options (e.g. the first and fourth options), without selecting any incorrect option (second option in this case), will result in +2 marks. Selecting only one of the three correct options (either first or third or fourth option), without selecting any incorrect option (second option in this case), will result in +2 marks. Selection of any correct options (in +1 marks. Selecting any incorrect option(s) (second option in this case), with or without selection of any correct option(s) will result in -2 marks.

### **Answering Section-1 Questions :**

- To select the option(s), using the mouse click on the corresponding button(s) of the option(s).
- To deselect chosen option(s), click on the button(s) of the chosen option(s) again or click on the **Clear Response** button to clear all the chosen options.
- To change the option(s) of a previously answered question, if required, first click on the **Clear Response** button to clear all the chosen options and then select the new option(s).
- To mark a question ONLY for review (i.e. without answering it), click on the Mark for Review & Next button.
- To mark a question for review (after answering it), click on **Mark for Review & Next button** answered question which is also marked for review will be evaluated.
- To save the answer, click on the **Save & Next** button the answered question will be evaluated.

### Instructions for Section-2 : Questions and Marking Scheme

### SECTION-2 (Maximum Marks : 24)

- This section contains EIGHT (08) questions. The answer to each question is NUMERICAL VALUE.
- For each question, enter the correct numerical value (in decimal notation, truncated/rounded-off to the second decimal place;
   e.g. 6.25, 7.00, -0.33, -0.30,,30.27, -127.30) using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.
- Answer to each question will be evaluated according to the following marking scheme :
  - Full Marks : +3 If ONLY the correct numerical value is entered as answer.
  - Zero Marks : **0** In all other cases.

### **Answering Section-2 Questions :**

- Using the attached computer mouse, click on numbers (and/or symbols) on the on-screen virtual numeric keypad to enter the numerical value as answer in the space provided for answer.
- To change the answer, if required, first click on the Clear Response button to clear the entered answer and then enter the new numerical value.
- To mark a question ONLY for review (i.e. answering it), click on **Mark for Review & Next button** the answered question which is also marked for review will be evaluated.
- To mark a question for review (after answering it), click Mark for Review & Next button the answered question which is also marked for review will be evaluated.
- To save the answer, click on the Save & Next button the answered question will be evaluated.

#### Instructions for Section-3 : Questions and Marking Scheme

#### SECTION-3 (Maximum Marks : 12)

- This section contains TWO (02) paragraphs. Based on each paragraph, there are TWO (02) questions.
- Each question has FOUR options. ONLY ONE of these four options corresponds to the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme :
  - Full Marks : +3 If ONLY the correct option is chosen.

Zero Marks : **0** If none of the options is chosen (i.e. the question is unanswered).

Negative Marks : -1 In all other cases.

#### **Answering Section-3 Questions :**

- To select an option, using the mouse click on the corresponding button of the option.
- To deselect the chosen answer, click on the button of the chosen option again or click on the Clear Response button.
- To change the chosen answer, click on the button of another option.
- To mark a question ONLY for review (i.e. without answering it), click on Mark for Review & Next button.
- To mark a question for review (after answering it), click on **Mark for Review & Next button** the answered which is also marked for review will be evaluated.
- To save the answer, click on the Save & Next button the answered question will be evaluated.

# PART I – PHYSICS

# **SECTION 1 (Maximum Marks : 24)**

- This section contains **SIX (06)** questions.
- Each question has **FOUR** options for correct answer(s). **ONE OR MORE THAN ONE** of these four option(s) is (are) correct option(s).
- For each question, choose the correct option(s) to answer the question.
- Answer to each question will be evaluated according to the following marking scheme :

Full Marks	:	+4	If only (all) the correct option(s) is (are) chosen.
Partial Marks	:	+3	If all the four option are correct but ONLY three options are chosen.
Partial Marks	:	+2	If three or more option are correct but ONLY two options are chosen, both of which are correct options.
Partial Marks	:	+1	If two or more option are correct but ONLY one option is chosen and it is a correct options.
Zero Marks Negative Marks	:	0 2	If none of the bubbles is chosen (i.e. the question is unanswered). In all other cases.

- For Example : If first, third and fourth are the ONLY three correct options for a question with second option being an incorrect option; selecting only all the three correct options will result in +4 marks. Selecting only two of the three correct options (e.g. the first and fourth options), without selecting any incorrect option (second option in this case), will result in +2 marks. Selecting only one of the three correct options (either first or third or fourth option), without selecting any incorrect option (second option in this case), will result in +1 marks. Selecting any incorrect option(s) (second option in this case), will result in +1 marks. Selecting any incorrect option(s) (second option in this case), will result in +1 marks. Selecting any incorrect option(s) (second option in this case), with or without selection of any correct option(s) will result in -2 marks.
- 1. The potential energy of a particle of mass *m* at a distance r from a fixed point 0 is given by  $V(r) = kr^2/2$ , where *k* is a positive constant of appropriate dimensions. This particle is moving in a circular orbit of radius *R* about the point 0. If *v* is the speed of the particle and *L* is the magnitude of its angular momentum about 0, which of the following statements is (are) true?

(A) 
$$v = \sqrt{\frac{k}{2m}}R$$
 (B)  $v = \sqrt{\frac{k}{m}}R$  (C)  $L = \sqrt{mk}R^2$  (D)  $L = \sqrt{\frac{mk}{2}}R^2$ 

Sol. (B), (C)

We know that, 
$$|F| = \frac{dv}{dr} = \frac{d}{dr} \left[\frac{kr^2}{2}\right] = kr$$

: Potential energy,  $V(r) = Kr^2/2$  given

For 
$$r = R$$
,  $F = kR$   
Also  $F = \frac{mv^2}{R}$  (circular motion)  
 $\therefore \frac{mv^2}{R} = kR \quad \therefore v = \sqrt{\frac{k}{m}} \times R$ 

Angular momentum  $L = mvR = m\left(\sqrt{\frac{k}{m}}R\right)R = \sqrt{km}R^2$ 

2. Consider a body of mass 1.0 kg at rest at the origin at time t = 0. A force  $\vec{F} = (\alpha t \hat{i} + \beta \hat{j})$  is applied on the body, where  $\alpha = 1.0 N s^{-1}$  and  $\beta = 1.0 N$ . The torque acting on the body about the origin at time t = 1.0 s is  $\vec{\tau}$ . Which of the following statements is (are) true?

(A) 
$$\left|\vec{\tau}\right| = \frac{1}{3}Nm$$

- (B) The torque  $\vec{\tau}$  is in the direction of the unit vector  $+\hat{k}$
- (C) The velocity of the body at t = 1 s is  $\vec{v} = \frac{1}{2} (\hat{i} + 2\hat{j}) m s^{-1}$
- (D) The magnitude of displacement of the body at t = 1s is  $\frac{1}{6}m$

Given 
$$\vec{F} = t\hat{i} + \hat{j}$$
  $\therefore \frac{md \vec{v}}{dt} = t\hat{i} + \hat{j}$   
 $\therefore d\vec{v} = t dt\hat{i} + dt\hat{j}$   $[\because m = 1]$   
 $\therefore \int_{0}^{v} d\vec{v} = \int_{0}^{t} t dt\hat{i} + \int_{0}^{t} dt\hat{j}$   
 $v = \frac{t^{2}}{2}\hat{i} + t\hat{j}$   
At  $t = 1s, v = \frac{1}{2}\hat{i} + \hat{j} = \frac{1}{2}(\hat{i} + 2\hat{j}) \text{ms}^{-1}$   
Further  $\frac{d\vec{r}}{dt} = \frac{t^{2}}{2}\hat{i} + t\hat{j}$   
 $\therefore d\vec{r} = \frac{t^{2}}{2} dt\hat{i} + t dt\hat{j}$   
 $\therefore \int_{0}^{r} d\vec{r} = \int_{0}^{t} \frac{t^{2}}{2} dt\hat{i} + \int_{0}^{t} t dt\hat{j} \Rightarrow \vec{r} = \frac{t^{3}}{6}\hat{i} + \frac{t^{2}}{2}\hat{j}$   
At  $t = 1, \vec{r} = \frac{1}{6}\hat{i} + \frac{1}{2}\hat{j}$   $\therefore |\vec{r}| = \sqrt{\frac{1}{36} + \frac{1}{4}} = \sqrt{\frac{10}{36}}$   
 $\vec{\tau} = \vec{r} \times \vec{F} = \left(\frac{1}{6}\hat{i} + \frac{1}{2}\hat{j}\right) \times (\hat{i} + \hat{j})$  (At  $t = 1s$ )  
 $= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{1}{6} & \frac{1}{2} & 0 \\ 1 & 1 & 0 \end{vmatrix} = \hat{i}(0 - 0) - \hat{j}(0 - 0) + \hat{k}\left(\frac{1}{6} - \frac{1}{2}\right) = -\frac{1}{3}\hat{k}$   
 $\therefore |\vec{r}| = \frac{1}{3}Nm$ 

- 3. A uniform capillary tube of inner radius *r* is dipped vertically into a beaker filled with water. The water rises to a height *h* in the capillary tube above the water surface in the beaker. The surface tension of water is  $\sigma$ . The angle of contact between water and the wall of the capillary tube is  $\theta$ . Ignore the mass of water in the meniscus. Which of the following statements is (are) true?
  - (A) For a given material of the capillary tube, h decreases with increase in r
  - (B) For a given material of the capillary tube, h is independent of  $\sigma$
  - (C) If this experiment is performed in a lift going up with a constant acceleration, then h decreases
  - (D) *h* is proportional to contact angle  $\theta$

We know that  $h = \frac{2\sigma\cos\theta}{r\rho g_{eff}}$ 

As 'r' increases, h decreases [all other parameter remaining constant]

Also  $h \propto \sigma$ 

Further if lift is going up with an acceleration 'a' then  $g_{eff} = g + a$ . As  $g_{eff}$  increases, 'h' decreases. Also h is not proportional to ' $\theta$ ' but  $h \propto \cos \theta$ 

- 4. In the figure below, the switches  $S_1$  and  $S_2$  are closed simultaneously at t = 0 and a current starts to flow in the circuit. Both the batteries have the same magnitude of the electromotive force (emf) and the polarities are as indicated in the figure. Ignore mutual inductance between the inductors. The current *I* in the middle wire reaches its maximum magnitude  $I_{max}$  at time  $t = \tau$ . Which of the following statements is (are) true?
- (A)  $I_{\max} = \frac{v}{2R}$ (B)  $I_{\max} = \frac{V}{4R}$ (C)  $\tau = \frac{L}{R} \ln 2$ (D)  $\tau = \frac{2L}{R} \ln 2$ Sol. **(B), (D)**

Here 
$$I + I_2 = I_1$$
  $\therefore I = I_1 - I_2$   
 $\therefore I = \frac{V}{R} \left[ 1 - e^{\frac{-Rt}{2L}} \right] - \frac{V}{R} \left[ 1 - e^{\frac{-Rt}{L}} \right] \Rightarrow I = \frac{V}{R} \left[ e^{\frac{-Rt}{L}} - e^{\frac{-Rt}{2L}} \right]$ 



For *I* to be maximum,  $\frac{dI}{dt} = 0$ 

$$\therefore \frac{V}{R} \left[ \frac{-R}{L} e^{\frac{-Rt}{L}} - \left(\frac{-R}{2L}\right) e^{\frac{-Rt}{2L}} \right] = 0$$
$$\therefore e^{\frac{-Rt}{2L}} = \frac{1}{2} \Longrightarrow \left(\frac{R}{2L}\right) t = \ln 2$$

$$\therefore t = \frac{2L}{R} \ln 2$$

This is the time when *I* is maximum

Further 
$$I_{\text{max}} = \frac{V}{R} \left[ e^{\frac{-R}{L} \left(\frac{2L}{R}\ell n 2\right)} - e^{\frac{-R}{2L} \left(\frac{2L}{R}\ell n 2\right)} \right] \Rightarrow I_{\text{max}} = \frac{V}{R} \left[ \frac{1}{4} - \frac{1}{2} \right]$$
  
$$\therefore I_{\text{max}} = \frac{V}{4R}$$

- 5. Two infinitely long straight wires lie in the xy plane along the lines  $x = \pm R$ . The wire located at x = +R carries a constant current  $I_1$  and the wire located at x = -R carries a constant current  $I_2$ . A circular loop of radius R is suspended with its centre at  $(0, 0, \sqrt{3}R)$  and in a plane parallel to the xy-plane. This loop carries a constant current I in the clockwise direction as seen from above the loop. The current in the wire is taken to be positive if it is in the  $+\hat{j}$  direction. Which of the following statements regarding the magnetic field  $\vec{B}$  is (are) true?
  - (A) If  $I_1 = I_2$ , then *B* cannot be equal to zero at the origin (0, 0, 0)
  - (B) If  $I_1 > 0$  and  $I_2 < 0$ , then  $\vec{B}$  can be equal to zero at the origin (0, 0, 0)
  - (C) If  $I_1 < 0$  and  $I_2 > 0$ , then  $\vec{B}$  can be equal to zero at the origin (0, 0, 0)
  - (D) If  $I_2 = I_2$ , then the z-component of the magnetic field at the centre of the loop is  $\left(-\frac{\mu_0 I}{2R}\right)$

# Sol. (A), (B), (D)

If  $I_1 = I_2$ , then the magnetic fields due to  $I_1$  and  $I_2$  at origin 'O' will cancel out each other. But the magnetic field at 'O' due to the circular loop will be present. Therefore 'A' is correct.

If  $I_1 > 0$  and  $I_2 < 0$ , then the magnetic field due to both current will be in +Z direction and add-up. The magnetic field due to current *I* will be in -Z direction and if its magnitude is equal to the combined magnitudes of  $I_1$  and  $I_2$ , then  $\vec{B}$  can be zero at the origin. Therefore option 'B' is correct.



If  $I_1 < 0$  and  $I_2 > 0$  then their net magnetic field at origin will be in -Z direction and the magnetic field due to *I* at origin will also be in -Z direction. Therefore  $\vec{B}$  at origin cannot be zero. Therefore 'C' is incorrect. If  $I_1 = I_2$  then the resultant of the magnetic field  $B_R$  at *P* (the centre of the circular loop) is along +*X* direction. Therefore the magnetic field at *P* is only due to the current *I* which is in -Z direction and is equal to  $\vec{B} = \frac{\mu_0 I}{2R} \left( -\hat{k} \right)$ .



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- 6. One mole of a monatomic ideal gas undergoes a cyclic process as shown in the figure (Where V is the volume and T is the temperature). Which of the statements below is (are) true?
  - (A) Process I is an isochoric process
  - (B) In process II, gas absorbs heat
  - (C) In process IV, gas releases heat
  - (D) Processes I and III are not isobaric

### Sol. (B), (C), (D)

In process I, volume is changing. Therefore it is not isochoric. Therefore 'A' is incorrect.

In process II,  $q = \Delta U + W$ .  $\Delta U = 0$  as temperature is constant. Therefore q = W. Here  $W = P(V_f - V_i)$  is positive therefore q is positive i.e., gas absorbs heat. Therefore 'B' is correct.

For process IV,  $q = \Delta U + W$ . Here  $\Delta U = 0$  and W is negative (volume decreases). Therefore q is negative i.e., gas releases heat. 'C' is correct.

For an isobaric process,  $V \propto T$  i.e., we will get a straight inclined line in *T*-*V* graph. Therefore I and II are NOT isobaric. 'D' is correct.

# **SECTION 2 (Maximum Marks : 24)**

- This section contains EIGHT (08) questions. The answer to each question is a NUMERICAL VALUE.
- For each question, enter the correct numerical value (in decimal notation, truncated/rounded-off to the **second decimal place;** e.g. 6.25, 7.00, -0.33, -.30, 30.27, -127.30) using the mouse and the onscreen virtual numeric keypad in the place designated to enter the answer.
- Answer to each question will be evaluated according to the following marking scheme :

Full Marks: +3If ONLY the correct numerical value is entered as answer.Zero Marks: 0In all other cases.

7. Two vectors  $\vec{A}$  and  $\vec{B}$  are defined as  $\vec{A} = a\hat{i}$  and  $\vec{B} = \alpha (\cos \omega \hat{i} + \sin \omega t \hat{j})$ , where *a* is a constant and  $\omega = \pi/6 rad s^{-1}$ . If  $|\vec{A} + \vec{B}| = \sqrt{3} |\vec{A} - \vec{B}|$  time  $t = \tau$  for the first time, the value of  $\tau$ , in *seconds*, is \_\_\_\_\_.

### Sol. (2.00)

$$\begin{vmatrix} \vec{A} + \vec{B} \end{vmatrix} = \sqrt{3} \begin{vmatrix} \vec{A} - \vec{B} \end{vmatrix}$$
  

$$\therefore |a\hat{i} + a\cos\omega t \,\hat{i} + a\sin\omega t \,\hat{j}| = \sqrt{3} |ai - a\cos\omega t \,\hat{i} - a\sin\omega t \,\hat{j}|$$
  

$$\Rightarrow |(1 + \cos\omega t)\hat{i} + \sin\omega t \,\hat{j}| = \sqrt{3} |(1 - \cos\omega t)\hat{i} - \sin\omega t \,\hat{j}|$$
  

$$\sqrt{2 + 2\cos\omega t} = \sqrt{3} \sqrt{2 - 2\cos\omega t}$$
  

$$\therefore 1 + \cos\omega t = 3(1 - \cos\omega t)$$
  

$$\Rightarrow 4\cos\omega t = 2 \quad \therefore \cos\omega t = \frac{1}{2} \quad \text{or,} \quad \omega t = \frac{\pi}{3}$$
  

$$\therefore \frac{\pi}{6} \times \tau = \frac{\pi}{3} \quad \therefore \tau = 2.00 \text{ seconds}$$



### JEE (Advanced) 2018

8. Two men are walking along a horizontal straight line in the same direction. The man in front walks at a speed  $1.0 m s^{-1}$  and the man behind walks at a speed  $2.0 m s^{-1}$ . A third man is standing at a height 12 m above the same horizontal line such that all three men are in a vertical plane. The two walking men are blowing identical whistles which emit a sound of frequency 1430 Hz. The speed of sound in air is  $330 m s^{-1}$ . At the instant, when the moving men are 10 m apart, the stationary man is equidistant from them. The frequency of beats in Hz, heard by the stationary man at this instant, is \_\_\_\_\_.

$$v_{A} = v \left[ \frac{v}{v - 2\cos\theta} \right]$$

$$v_{B} = v \left[ \frac{v}{v + 1\cos\theta} \right]$$

$$\therefore \text{ Beat frequency} = v \left[ \frac{v}{v - 2\cos\theta} \right] - v \left[ \frac{v}{v + \cos\theta} \right]$$

$$= v v \left[ \frac{1}{v - 2\cos\theta} - \frac{1}{v + \cos\theta} \right] = 1430 \times 330 \left[ \frac{1}{330 - 2 \times \frac{5}{13}} - \frac{1}{330 + \frac{5}{13}} \right]$$

$$= 1430 \times 330 \times 13 \left[ \frac{1}{330 \times 13 - 10} - \frac{1}{330 \times 13 + 5} \right] = 1430 \times 330 \times 13 \left[ \frac{1}{4280} - \frac{1}{4295} \right] \approx 5 \text{ Hz}$$

- 9. A ring and a disc are initially at rest, side by side, at the top of an inclined plane which makes an angle 60° with the horizontal. They start to roll without slipping at the same instant of time along the shortest path. If the time difference between their reaching the ground is  $(2-\sqrt{3})/\sqrt{10} s$ , then the height of the top of the inclined plane, in *metres*, is \_\_\_\_\_ Take  $g = 10 m s^{-2}$
- Sol. (0.75)

The time taken to reach the ground is given by  $t = \frac{1}{\sin \theta} \sqrt{\frac{2h}{g} \left(1 + \frac{I_c}{MR^2}\right)}$ 

For ring 
$$t_1 = \frac{1}{\sin 60^\circ} \sqrt{\frac{2h}{g} \left(1 + \frac{MR^2}{MR^2}\right)} = \frac{4}{\sqrt{3}} \sqrt{\frac{h}{g}}$$
  
For disc  $t_2 = \frac{1}{\sin 60^\circ} \sqrt{\frac{2h}{g} \left(1 + \frac{1}{2} \frac{MR^2}{MR^2}\right)} = \frac{2}{\sqrt{3}} \sqrt{\frac{3h}{g}}$ 

Given 
$$t_1 - t_2 = \frac{2 - \sqrt{3}}{\sqrt{10}}$$
  
 $\therefore \frac{4}{\sqrt{3}} \sqrt{\frac{h}{g}} - \frac{2}{\sqrt{3}} \sqrt{\frac{3h}{g}} = \frac{2 - \sqrt{3}}{\sqrt{10}}$   
 $2\sqrt{\frac{h}{10}} - \sqrt{\frac{3h}{10}} = \frac{(2 - \sqrt{3})}{\sqrt{10}} \left(\frac{\sqrt{3}}{2}\right)$ 

$$2\sqrt{h} - \sqrt{3h} = \sqrt{3} - \frac{3}{2}$$
$$\sqrt{h} (2 - 1.732) = 1.732 - 1.5 \qquad \therefore \sqrt{h} = \frac{0.232}{0.268}$$
$$\therefore h \approx 0.75 \,\mathrm{m}$$

10. A spring-block system is resting on a frictionless floor as shown in the figure. The spring constant is 2.0  $N m^{-1}$  and the mass of the block is 2.0 kg. Ignore the mass of the spring. Initially the spring is in an unscratched condition. Another block of mass 1.0 kg moving with a speed of 2.0  $m s^{-1}$  collides elastically with the first block. The collision is such that the 2.0 kg block does not hit the wall. The distance, in *metres*, between the two blocks when the spring returns to its unstretched position for the first time after the collision is d = 1.



$$v_{1} = \frac{(m_{1} - m_{2})u_{1}}{m_{1} + m_{2}} + \frac{2m_{2}u_{2}}{m_{1} + m_{2}} = \frac{(1 - 2)2}{1 + 2} = \frac{-2}{3} \text{ ms}^{-1}$$

$$v_{2} = \frac{(m_{2} - m_{1})u_{2}}{m_{1} + m_{2}} + \frac{2m_{1}u_{1}}{m_{1} + m_{2}} = \frac{2 \times 1 \times 2}{1 + 2} = \frac{4}{3} \text{ ms}^{-1}$$

$$u_{1} = 2 \text{ ms}^{-1}$$

$$u_{2} = 0$$

$$2 \text{ kg}$$

The time period of mass 2 kg after attaining velocity is

$$T = 2\pi \sqrt{\frac{m}{k}} = 2\pi \sqrt{\frac{2}{2}} = 2\pi$$

Therefore the time taken to return the original position by 2 kg mass is  $\pi$  sec.

- :. Distance between the two blocks  $=\frac{2}{3} \times \pi = \frac{2}{3} \times \frac{22}{7} = 2.09 \text{ m}$
- 11. Three identical capacitors  $C_1, C_2$  and  $C_3$  have a capacitance of 1.0  $\mu F$  each and they are uncharged initially. They are connected in a circuit as shown in the figure and  $C_1$  is then filled completely with a dielectric material of relative permittivity  $\in_r$ . The cell electromotive force (emf) $V_0 = 8V$ . First the switch  $S_1$  is closed while the switch  $S_2$  is kept open. When the capacitor  $C_3$  is fully charged,  $S_1$  is opened and  $S_2$  is closed simultaneously. When all the capacitors reach equilibrium, the charge on  $C_3$  is found to be  $5\mu C$ . The value of  $\in_r =$

$$\in_r = \_$$

# Sol. (1.50)

### Initially

The charge on  $C_3$  is  $q_3 = C_3 V = 1 \times 8 \mu C = 8 \mu C$ 





### Finally

As the charge on  $C_3$  is found to be  $5\mu C$  therefore charges on  $C_1$  and  $C_2$  are  $3\mu C$  each

Applying Kirchhoff loop law



12. In the *xy*-plane, the region y > 0 has a uniform magnetic field  $B_1\hat{k}$  and the region y < 0 has another uniform magnetic field  $B_2\hat{k}$ . A positively charged particle is projected from the origin along the positive *y*-axis with speed  $v_0 = \pi ms^{-1}$  at t = 0, as shown in the figure. Neglect gravity in this problem. Let t = T be the time when the particle crosses the *x*-axis from below for the first time. If  $B_2 = 4B_1$ , the average speed of the particle, in  $ms^{-1}$ , along the *x*-axis in the time interval *T* is \_\_\_\_\_.



### Sol. (2.00)

Average speed along X-axis 
$$= \frac{D_1 + D_2}{t_1 + t_2} = \frac{2(R_1 + R_2)}{t_1 + t_2} = 2 \left[ \frac{\frac{mv_0}{qB_1} + \frac{mv_0}{q(4B_1)}}{\frac{\pi m}{qB_1} + \frac{\pi m}{q(4B_1)}} \right]$$

13. Sunlight of intensity  $1.3 kW m^{-2}$  is incident normally on a thin convex lens of focal length 20 *cm*. Ignore the energy loss of light due to the lens and assume that the lens aperture size is much smaller than its focal length. The average intensity of light, in  $kW m^{-2}$ , at a distance 22 *cm* from the lens on the other side is \_\_\_\_\_.

### Sol. (130.00)

 $\triangle AFB$  and  $\triangle CFD$  are similar  $\frac{d}{D} = \frac{2}{20} = \frac{1}{10}$   $\therefore \text{ Ratio of area} = \frac{d^2}{D^2} = \frac{1}{100}$ 

As there is no energy loss

- $\therefore \text{ Average intensity of light at a distance } 22 \text{ cm} = \frac{1.3 \times \pi D^2 / 4}{\pi d^2 / 4} = 1.3 \times 100 = 130.00 \text{ kWm}^{-2}$
- 14 Two conducting cylinders of equal length but different radii are connected in series between two heat baths kept at temperatures  $T_1 = 300 k$  and  $T_2 = 100 k$ , as shown in the figure. The radius of the bigger cylinder is twice that of the smaller one and the thermal conductivities of the materials of the smaller and the larger cylinders are  $k_1$  and  $k_2$  respectively. If the temperature at the junction of the two cylinders in the steady state is 200 k, then  $K_1/K_2 =$  \_\_\_\_\_\_.



# Sol. (4.00)

The intermediate temperature is given by the formula

$$T = \frac{\frac{k_1 A_1 T_1}{l_1} + \frac{k_2 A_2 T_2}{l_2}}{\frac{k_1 A_1}{l_1} + \frac{k_2 A_2}{l_2}}$$
  
Here,  $T = 200 k$ ,  $T_1 = 300 k$ ,  $T_2 = 100 k$   
 $l_1 = l_2$  and  $A_1 = \pi r^2$ ,  $A_2 = 4\pi r^2$   
 $\therefore 200 = \frac{k_1 \pi r^2 \times 300 + k_2 \pi (4r^2) 100}{k_1 \pi r^2 + k_2 \pi (4r^2)}$   
 $\therefore 200 = \frac{300 k_1 + 400 k_2}{k_1 + 4k_2}$ 

$$\therefore \quad 200 \, k_1 + 800 k_2 = 300 k_1 + 400 k_2 \implies 400 \, k_2 = 100 \, k_1 \quad \therefore \quad \frac{k_1}{k_2} = 4.00$$

# **SECTION 3 (Maximum Marks : 12)**

- This section contains TWO (02) paragraphs. Based on each paragraph, there are TWO (02) questions.
- Each question has FOUR options. ONLY ONE of these four options corresponds to the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme :

Full Marks	:	+3	If ONLY the correct option is chosen.
Zero Marks	:	0	If none of the option is chosen (i.e. the question is unanswered)
Negative Marks	:	-1	In all other cases.

# PARAGRAPH "X"

In electromagnetic theory, the electric and magnetic phenomena are related to each other. Therefore, the dimensions of electric and magnetic quantities must also be related to each other. In the questions below, [E] and [B] stand for dimensions of electric and magnetic fields respectively, while  $[\in_0]$  and  $[\mu_0]$  stand for dimensions of the permittivity and permeability of free space respectively. [L] and [T] are dimensions of length and time respectively. All the quantities are given in SI units.

### (There are two questions based on PARAGRAPH "X", the question given below is one of them)

- 15. The relation between [E] and [B] is
  - (A) [E] = [B][L][T](C)  $[E] = [B] [L] [T]^{-1}$

Sol. (C)

We know that,  $C = \frac{E}{R}$  where C = speed of light

$$\therefore E = CB = LT^{-1}B$$

- The relation between  $[\in_0]$  and  $[\mu_0]$  is 16.
  - (B)  $[\mu_{0}] = [\epsilon_{0}] [L]^{-2} [T]^{2}$ (A)  $[\mu_0] = [\epsilon_0] [L]^2 [T]^{-2}$ (C)  $[\mu_0] = [\epsilon_0]^{-1} [L]^2 [T]^{-2}$

Sol. (D)

We know that

$$C = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \quad \therefore \quad C^2 = \frac{1}{\mu_0 \varepsilon_0}$$
$$\therefore \quad \mu_0 = \varepsilon_0^{-1} L^{-2} T^2$$

(D) 
$$[\mu_0] = [\epsilon_0]^{-1} [L]^{-2} [T]^2$$

(B)  $[E] = [B] [L]^{-1} [T]$ 

(D)  $[E] = [B] [L]^{-1} [T]^{-1}$ 

### PARAGRAPH "A"

If the measurement errors in all the independent quantities are known, then it is possible to determine the error in any dependent quantity. This is done by the use of series expansion and truncating the expansion at the first power of the error. For example, consider the relation z = x/y. If the errors in x, y and z are  $\Delta x$ ,  $\Delta y$  and  $\Delta z$ , respectively, then

$$z \pm \Delta x = \frac{x \pm \Delta x}{y \pm \Delta y} = \frac{x}{y} \left( 1 \pm \frac{\Delta x}{x} \right) \left( 1 \pm \frac{\Delta y}{y} \right)^{-1}$$

The series expansion for  $\left(1\pm\frac{\Delta x}{y}\right)^{-1}$ , to first power in  $\Delta y/y$ , is  $1\mp \left(\Delta y/y\right)$ . The relative errors in independent variables are always added. So the error in z will be

$$\Delta z = z \left( \frac{\Delta x}{x} + \frac{\Delta y}{y} \right).$$

The above derivation makes the assumption that  $\Delta x/x \ll 1$ ,  $\Delta y/y \ll 1$ . Therefore; the higher powers of these quantities are neglected.

# (There are two questions based on PARAGRAPH "A", the question given below is one of them)

17. Consider the ratio  $r = \frac{(1-a)}{(1+a)}$  to be determined by measuring a dimensionless quantity *a*. If the error in the measurement of *a* is  $\Delta a (\Delta a / \alpha << 1)$ , then what is the error  $\Delta r$  in determining r?

 $\Delta a$   $\Delta a$   $\Delta a$   $\Delta a$   $\Delta a$   $\Delta a$   $\Delta a$ 

(A) 
$$\frac{24}{(1+a)^2}$$
 (B)  $\frac{244}{(1+a)^2}$  (C)  $\frac{244}{(1-a^2)}$  (D)  $\frac{2444}{(1-a^2)}$ 

Sol. (B)

$$r = \frac{1-a}{1+a} \therefore \frac{dr}{da} = \frac{(1+a)(-1)-(1-a)}{(1+a)^2} = \frac{-2}{(1+a)^2}$$
$$\therefore |\Delta r| = \frac{2\Delta a}{(1+a)^2}$$

18. In an experiment the initial number of radioactive nuclei is 3000. It is found that  $1000 \pm 40$  nuclei decayed in the first 1.0 s. For  $|x| << 1, \ln(1+x) = x$  up to first power in x. The error  $\Delta\lambda$ , in the determination of the decay constant  $\lambda$ , in  $s^{-1}$  is

(A) 0.04 (B) 0.03 (C) 0.02 (D) 0.01

Sol. (C)

We know that,  $N = N_0 e^{-\lambda t}$ 

Taking log on both sides  $\log_e N = \log_e N_0 - \lambda t$  differentiating with respect to ' $\lambda$ ' we get

$$\frac{1}{N}\frac{dN}{d\lambda} = 0 - t$$
  

$$\therefore |d\lambda| = \frac{dN}{tN} = \frac{40}{1 \times 2000} = 0.02 \quad [\because N = 3000 - 1000 = 2000]$$

# PART II – CHEMISTRY

# **SECTION 1 (Maximum Marks : 24)**

- This section contains **SIX (06)** questions.
- Each question has **FOUR** options for correct answer(s). **ONE OR MORE THAN ONE** of these four option(s) is (are) correct option(s).
- For each question, choose the correct option(s) to answer the question.
- Answer to each question will be evaluated according to the following marking scheme :

Full Marks	:	+4	If only (all) the correct option(s) is (are) chosen.
Partial Marks	:	+3	If all the four option are correct but ONLY three options are chosen.
Partial Marks	:	+2	If three or more option are correct but ONLY two options are chosen, both of which are correct options.
Partial Marks	:	+1	If two or more option are correct but ONLY one option is chosen and it is a correct options.
Zero Marks	:	0	If none of the bubbles is chosen (i.e. the question is unanswered).
Negative Marks	:	-2	In all other cases.

- For Example : If first, third and fourth are the ONLY three correct options for a question with second option being an incorrect option; selecting only all the three correct options will result in +4 marks. Selecting only two of the three correct options (e.g. the first and fourth options), without selecting any incorrect option (second option in this case), will result in +2 marks. Selecting only one of the three correct options (either first or third or fourth option), without selecting any incorrect option in this case), will result in +1 marks. Selecting any incorrect option(s) (second option in this case), will result in -2 marks.
- 19. The compound(s) which generate(s) N<sub>2</sub> gas upon thermal decomposition below 300°C is (are)

(A) 
$$NH_4NO_3$$
 (B)  $(NH_4)_2 Cr_2O_7$  (C)  $Ba(N_3)_2$  (D)  $Mg_3N_2$ 

(A) NH<sub>4</sub>NO<sub>3</sub> 
$$\xrightarrow{\text{below 300°C}}$$
 N<sub>2</sub>O + 2H<sub>2</sub>O  
N<sub>2</sub>O  $\xrightarrow{\text{above 600°C}}$  N<sub>2</sub> + O<sub>2</sub>  
(B) (NH<sub>4</sub>)<sub>2</sub>Cr<sub>2</sub>O<sub>7</sub>  $\xrightarrow{\Delta}$  N<sub>2</sub> + Cr<sub>2</sub>O<sub>3</sub> + 4H<sub>2</sub>O

- (C) Ba(N<sub>3</sub>)<sub>2</sub>  $\xrightarrow{\Delta}$  Ba + 3N<sub>2</sub>
- (D) Mg<sub>3</sub>N<sub>2</sub>  $\longrightarrow$  3Mg + N<sub>2</sub>

Hence only (NH<sub>4</sub>)<sub>2</sub>Cr<sub>2</sub>O<sub>7</sub> and Ba(N<sub>3</sub>)<sub>2</sub> can provide N<sub>2</sub> gas on heating below 300°C (B,C)

- 20. The correct statement(s) regarding the binary transition metal carbonyl compounds is (are) (Atomic numbers: Fe = 26, Ni = 28)
  - (A) Total number of valence shell electrons at metal centre in Fe (CO)<sub>5</sub> or Ni(CO)<sub>4</sub> is 16
  - (B) These are predominantly low spin in nature
  - (C) Metal carbon bond strengthens when the oxidation state of the metal is lowered
  - (D) The carbonyl C–O bond weakens when the oxidation state of the metal is increased

Sol. (B), (C)

- (A) [(Fe(CO<sub>5</sub>)] & [Ni(CO)<sub>4</sub>] complexes have 18-electrons in their valence shell.
- (B) Due to strong ligand field, carbonyl complexes are predominantly low spin complexes.
- (C) As electron density increases on metals (with lowering oxidation state on metals), the extent of synergic bonding increases. Hence M-C bond strength increases
- (D) While positive charge on metals increases and the extent of synergic bond decreases and hence C–O bond becomes stronger.
- 21. Based on the compounds of group 15 elements, the correct statement(s) is (are)
  - (A)  $Bi_2O_5$  is more basic than  $N_2O_5$
  - (B)NF<sub>3</sub> is more covalent than BiF<sub>3</sub>
  - (C) PH<sub>3</sub> boils at lower temperature than NH<sub>3</sub>
  - (D) The N-N single bond is stronger than the P-P single bond

Sol. (A), (B), (C)

- (A) Basic character of oxides increase on moving down the group therefore  $Bi_2O_5$  is more basic than  $N_2O_5$ .
- (B) Covalent nature depends on electronegativity difference between bonded atoms. In NF<sub>3</sub>, N and F are non-metals but in BiF<sub>3</sub>, Bi is metal while F is non metal therefore NF<sub>3</sub> is more covalent than BiF<sub>3</sub>.
- (C) In PH<sub>3</sub> hydrogen bonding is absent but in NH<sub>3</sub> hydrogen bonding is present, therefore PH<sub>3</sub> boils at lower temperature than NH<sub>3</sub>.
- (D) Due to small size in N–N single bond 1.p. 1.p. repulsion is more than P–P single bond therefore N–N single bond is weaker than the P–P single bond.
- 22. In the following reaction sequence, the correct structure(s) of X is (are)

 $X = \frac{1) \operatorname{PBr}_3, \operatorname{Et}_2 O}{2) \operatorname{Nal}, \operatorname{Me}_2 CO}$ 

3) NaN<sub>3</sub>, HCONMe<sub>2</sub>



enantiomerically pure

(A) Me OH









23. The reaction(s) leading to the formation of 1, 3, 5-trimethylbenzene is (are)



Sol. (A), (B), (D)



24. A reversible cyclic process for an ideal gas is shown below. Here, *P*, *V*, and *T* are pressure, volume and temperature, respectively. The thermodynamic parameters *q*, *w*, *H* and *U* are heat, work, enthalpy and internal energy, respectively.



The correct option(s) is (are)

(A) 
$$q_{AC} = \Delta U_{BC}$$
 and  $W_{AB} = P_2 (V_2 - V_1)$   
(B)  $w_{BC} = P_2 (V_2 - V_1)$  and  $q_{BC} = \Delta H_{AC}$   
(C)  $\Delta H_{CA} < \Delta U_{CA}$  and  $q_{AC} = \Delta U_{BC}$   
(D)  $q_{BC} = \Delta H_{AC}$  and  $\Delta H_{CA} > \Delta U_{CA}$ 

Sol. (B), (C)

- $A C \Rightarrow$  isochoric process
- $A B \Rightarrow$  isothermal process
- $B-C \Rightarrow$  isobaric process

$$\Rightarrow q_{AC} = \Delta U_{AC} = nC_{V,m}(T_2 - T_1) = \Delta U_{BC}$$

$$\Rightarrow W_{AB} = -nRT_1 \ln\left(\frac{V_2}{V_1}\right) \Rightarrow W_{BC} = -P_2(V_1 - V_2) = P_2(V_2 - V_1)$$

 $\Rightarrow q_{BC} = \Delta H_{BC} = nC_{P,m} (T_2 - T_1) = \Delta H_{AC} \Rightarrow \Delta H_{CA} = nC_{P,m} (T_1 - T_2)$ 

$$\Rightarrow \Delta U_{CA} = nC_{V,m}(T_1 - T_2)$$

 $\Delta H_{CA} \leq \Delta U_{CA}$  since both are negative (T<sub>1</sub> < T<sub>2</sub>)

# **SECTION 2 (Maximum Marks : 24)**

- This section contains EIGHT (08) questions. The answer to each question is a NUMERICAL VALUE.
- For each question, enter the correct numerical value (in decimal notation, truncated/rounded-off to the **second decimal place;** e.g. 6.25, 7.00, -0.33, -.30, 30.27, -127.30) using the mouse and the onscreen virtual numeric keypad in the place designated to enter the answer.
- Answer to each question will be evaluated according to the following marking scheme :

Full Marks	:	+3	If ONLY the correct numerical value is entered as answer.
Zero Marks	:	0	In all other cases.

25. Among the species given below, the total number of diamagnetic species is \_\_\_\_\_\_ H atom, NO<sub>2</sub> monomer,  $O_2^-$  (superoxide), dimeric sulphur in vapour phase, Mn<sub>3</sub>O<sub>4</sub>, (NH<sub>4</sub>)<sub>2</sub>[FeCl<sub>4</sub>], (NH<sub>4</sub>)<sub>2</sub>[NiCl<sub>4</sub>], K<sub>2</sub>MnO<sub>4</sub>, K<sub>2</sub>CrO<sub>4</sub>

Sol. (1)

•	H-atom $=1s^1$	Paramagnetic
•	$NO_2 = $ N odd electron species	Paramagnetic
•	$O_2^-$ (superoxide) = One unpaired electron in $\pi^*M.O.$	Paramagnetic

•  $S_2$  (in vapour phase) = same as  $O_2$ , two unpaired e<sup>-</sup>s are present in  $\pi^*M.O.$  Paramagnetic

• 
$$Mn_3O_4 = 2 MnO_1 MnO_2$$

- $(NH_4)_2[FeCl_4] = Fe^{2+} = 3d^64s^0$
- $(NH_4)_2[NiCl_4] = Ni = 3d^84s^2$   $Ni^{2+} = 3d^84s^0$

• 
$$K_2MnO_4 = 2K^+ \begin{bmatrix} O^-\\ Mn\\ O & O^- \end{bmatrix}$$
,  $Mn^{6+} = [Ar]3d^1$  Paramagnetic  
•  $K_2CrO_4 = 2K^+ \begin{bmatrix} O\\ \\ \\ O & O^- \end{bmatrix}$ ,  $Cr^{6+} = [Ar]3d^0$  Diamagnetic

26. The ammonia prepared by treating ammonium sulphate with calcium hydroxide is completely used by NiCl<sub>2</sub>.6H<sub>2</sub>O to form a stable coordination compound. Assume that both the reactions are 100% complete. If 1584 g of ammonium sulphate and 952 g of NiCl<sub>2</sub>.6H<sub>2</sub>O are used in the preparation, the combined weight (in grams) of gypsum and the nickel-ammonia coordination compound thus produced is \_\_\_\_\_.

(Atomic weights in g mol<sup>-1</sup>: H = 1, N = 14, O = 16, S = 32, Cl = 35.5, Ca = 40, Ni = 59)

Sol. (2992)

$$(\mathrm{NH}_{4})_{2} \operatorname{SO}_{4} + \operatorname{Ca}(\mathrm{OH})_{2} \rightarrow \operatorname{CaSO}_{4} \cdot 2\mathrm{H}_{2}\mathrm{O} + 2\mathrm{NH}_{3}_{24 \text{ mole}}$$

$$\underset{\substack{952g=4 \text{ mole}}{\text{ mole}}}{\text{NiCl}_2.6H_2O+6NH_3} \rightarrow \underset{\substack{(M=232)\\4 \text{ mole}}{\text{ mole}} \text{Ni(NH_3)}_6 ]Cl_2 + 6H_2O$$

Total mass =  $12 \times 172 + 4 \times 232 = 2992$  g

- 27. Consider an ionic solid **MX** with NaCl structure. Construct a new structure (**Z**) whose unit cell is constructed from the unit cell of **MX** following the sequential instructions given below. Neglect the charge balance.
  - (i) Remove all the anions (X) except the central one
  - (ii)Replace all the face centered cations (M) by anions (X)
  - (iii) Remove all the corner cations (M)
  - (iv) Replace the central anion (X) with cation (M)

The value of 
$$\left(\frac{number \ of \ anions}{number \ of \ cations}\right)$$
 in Z is\_\_\_\_\_.

### Sol. (3)

As per given information, cations form fee lattice and anions occupy all the octahedral voids.

Formula MX

So	$M^+$	$X^{-}$
	4 ions	4 ions
After step I	4 ions	1 ion
After step II	1 ion	4 ions
After step III	0 ion	4 ions
After step IV	1 ion	3 ions

So ratio of  $\frac{\text{No. of anions}}{\text{No. of cations}} = \frac{3}{1}$ 

28. F or the electrochemical cell,

 $Mg(s) \mid Mg^{2^+} \left(aq, 1 \text{ M}\right) \parallel Cu^{2^+} \left(aq, 1 \text{ M}\right) \mid Cu(s)$ 

the standard emf of the cell is 2.70 V at 300 K. When the concentration of  $Mg^{2+}$  is changed to x M, the cell potential changes to 2.67 V at 300 K. The value of x is

(given,  $\frac{F}{R} = 11500 \text{ K V}^{-1}$ , where F is the Faraday constant and R is the gas constant,  $\ln(10) = 2.30$ )

Sol. (10)

$$Mg(s) \longrightarrow Mg^{2+}(aq) + 2e^{-}$$

$$Cu^{2+}(aq) + 2e^{-} \longrightarrow Cu(s)$$

$$Mg(s) + Cu^{2+}(aq) \longrightarrow Mg^{2+}(aq) + Cu(s)$$

$$E_{cell} = E_{cell}^{\circ} - \frac{RT}{nF} \ln x$$

$$E = 2.67 = 2.7 - \frac{RT}{nF} \ln \frac{x}{1}$$

$$0.03 = \frac{300}{2 \times 11500} \ln x$$

$$2.3 = \ln x$$

$$x = 10$$

29. A closed tank has two compartments **A** and **B**, both filled with oxygen (assumed to be ideal gas). The partition separating the two compartments is fixed and is a perfect heat insulator (Figure 1). If the old partition is replaced by a new partition which can slide and conduct heat but does **NOT** allow the gas to leak across (Figure 2), the volume (in m<sup>3</sup>) of the compartment **A** after the system attains equilibrium is \_\_\_\_\_.



 $\frac{3}{300R}$ 

$$n_1 = \frac{5}{400R}$$
  $n_2 =$ 

Sol. (2.22)

 $P_1 = 5$  bar

 $V_1 = 1 m^3$ 

 $T_1 = 400 \, K$ 

Let volume be (V+x) V = (3-x)

$$\frac{P_A}{T_A} = \frac{P_B}{T_B} \implies \frac{n_{b_1} \times R}{V_{b_1}} = \frac{n_{b_2} \times R}{V_{b_2}} \implies \frac{5}{400(4 \times x)} = \frac{3}{300R(3-x)} \implies 5(3-x) = 4 + 4x \implies x = \frac{11}{9}$$
$$V = 1 + x = 1 + \frac{11}{9} = \left(\frac{20}{9}\right) = 2.22$$

30. Liquids **A** and **B** form ideal solution over the entire range of composition. At temperature T, equimolar binary solution of liquids **A** and **B** has vapour pressure 45 Torr. At the same temperature, a new solution of **A** and **B** having mole fractions  $x_A$  and  $x_B$ , respectively, has vapour pressure of 22.5 Torr. The value of  $x_A/x_B$  in the new solution is \_\_\_\_\_.

(given that the vapour pressure of pure liquid A is 20 Torr at temperature T).

$$P_{T} = p_{A}^{\circ} X_{A} + p_{B}^{\circ} X_{B} \qquad 45 = 20(0.5) + p_{B}^{\circ}(0.5)$$

$$p_{B}^{\circ} = 70 \quad 22.5 = 20X_{A} + 70(1 - X_{A})$$

$$50X_{A} = 47.5$$

$$X_{A} = \frac{4.75}{5} = 0.95 \quad X_{B} = 0.05 \quad \frac{X_{A}}{X_{B}} = 19$$

- 31. The solubility of a salt of weak acid (**AB**) at pH 3 is  $\mathbf{Y} \times 10^{-3} \text{ mol } \text{L}^{-1}$ . The value of  $\mathbf{Y}$  is \_\_\_\_\_. (Given that the value of solubility product of **AB** ( $K_{sp}$ ) = 2× 10<sup>-10</sup> and the value of ionization constant of HB ( $K_a$ ) = 1×10<sup>-8</sup>)
- Sol. (4.47)

$$S = \sqrt{K_{sp} \left(\frac{\left[H^{+}\right]}{K_{a}} + 1\right)} = \sqrt{20 \times 10^{-10} \left(\frac{10^{-3}}{10^{-8}} + 1\right)} \approx \sqrt{2 \times 10^{-5}} = 4.47 \times 10^{-3} M$$

32. The plot given below shows P-T curves (where *P* is the pressure and *T* is the temperature) for two solvents **X** and **Y** and isomolal solutions of NaCl in these solvents. NaCl completely dissociates in both the solvents.



On addition of equal number of moles of a non-volatile solute S in equal amount (in kg) of these solvents, the elevation of boiling point of solvent X is three times that of solvent Y. Solute S is known to undergo dimerization in these solvents. If the degree of dimerization is 0.7 in solvent Y, the degree of dimerization in solvent X is \_\_\_\_\_.

# Sol. (0.05)

From graph

For solvent 'X'  $\Delta T_{b(x)} = 362 - 360 = 2$ 

 $\Delta T_{b(x)} = m_{NaCl} \times K_{b(x)} \qquad \dots (1)$ 

For solvent 'Y'  $\Delta T_{b(y)} = 368 - 367 = 1$ 

$$\Delta T_{b(y)} = m_{NaCl} \times K_{b(y)} \qquad \dots (2)$$

Dividing equation (1) by (2)

$$\Rightarrow \frac{K_{b(x)}}{K_{b(y)}} = 2$$

For solute S

Given solute S dimerizes in solvent. Hence,

$$2(S) \rightarrow S_{2} \quad i = (1 - \alpha/2)$$

$$\Delta T_{b(x)(s)} = \left(1 - \frac{\alpha_{1}}{2}\right) K_{b(x)} \quad \Delta T_{b(y)(s)} = \left(1 - \frac{\alpha_{2}}{2}\right) K_{b(y)}$$
Given  $\Delta T_{b(x)(s)} = 3\Delta T_{b(y)(s)}$ 

$$\left(1 - \frac{\alpha_1}{2}\right) \mathbf{K}_{\mathbf{b}(\mathbf{x})} = 3 \times \left(1 - \frac{\alpha_2}{2}\right) \times \mathbf{K}_{(\mathbf{b})(\mathbf{y})} \quad 2\left(1 - \frac{\alpha_1}{2}\right) = 3\left(1 - \frac{\alpha_2}{2}\right)$$

$$\alpha_{2} = 0.7$$

so  $\alpha_1 = 0.05$ 

# **SECTION 3 (Maximum Marks : 12)**

- This section contains TWO (02) paragraphs. Based on each paragraph, there are TWO (02) questions.
- Each question has FOUR options. ONLY ONE of these four options corresponds to the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme :

Full Marks	:	+3	If ONLY the correct option is chosen.
Zero Marks	:	0	If none of the option is chosen (i.e. the question is unanswered).
Negative Marks	:	-1	In all other cases.

# PARAGRAPH "X"

Treatment of benzene with CO/HCl in the presence of anhydrous  $AlCl_3/CuCl$  followed by reaction with  $Ac_2O/NaOAc$  gives compound **X** as the major product. Compound **X** upon reaction with  $Br_2/Na_2CO_3$ , followed by heating at 473 K with moist KOH furnishes **Y** as the major product. Reaction of **X** with H2/Pd-C, followed by H3PO4 treatment gives **Z** as the major product.

### (There are two questions based on PARAGRAPH "X", the question given below is one of them)

33. The compound **Y** is





An organic acid  $\mathbf{P}$  (C<sub>11</sub>H<sub>12</sub>O<sub>2</sub>) can easily be oxidized to a dibasic acid which reacts with ethyleneglycol to produce a polymer dacron. Upon ozonolysis,  $\mathbf{P}$  gives an aliphatic ketone as one of the products.  $\mathbf{P}$  undergoes the following reaction sequences to furnish  $\mathbf{R}$  *via*  $\mathbf{Q}$ . The compound  $\mathbf{P}$  also undergoes another set of reactions to produce  $\mathbf{S}$ .

$$S \xrightarrow{1) H_2/Pd-C} 2) NH_3/\Delta 3) Br_2/NaOH 
S \xrightarrow{4) CHCl_3, KOH, \Delta 5) H_2/Pd-C P \\
\xrightarrow{1) H_2/Pd-C} P \xrightarrow{1) H_2/Pd-C} Q \xrightarrow{1) HCl} 2) SOCl_2 \\
\xrightarrow{1) H_2/Pd-C} Q \xrightarrow{2) Mg/Et_2O} Q \xrightarrow{3) CO_2 (dry ice)} R$$

ŕ

# (There are two questions based on PARAGRAPH "X", the question given below is one of them)

35. The compound R is



### Paper 1

# **PART III – MATHEMATICS**

# **SECTION 1 (Maximum Marks : 24)**

- This section contains **SIX (06)** questions.
- Each question has **FOUR** options for correct answer(s). **ONE OR MORE THAN ONE** of these four option(s) is (are) correct option(s).
- For each question, choose the correct option(s) to answer the question.
- Answer to each question will be evaluated according to the following marking scheme :

Full Marks	:	+4	If only (all) the correct option(s) is (are) chosen.
Partial Marks	:	+3	If all the four option are correct but ONLY three options are chosen.
Partial Marks	:	+2	If three or more option are correct but ONLY two options are chosen, both of which are correct options.
Partial Marks	:	+1	If two or more option are correct but ONLY one option is chosen and it is a correct options.
Zero Marks	:	0	If none of the bubbles is chosen (i.e. the question is unanswered).
Negative Marks	:	-2	In all other cases.

- For Example : If first, third and fourth are the ONLY three correct options for a question with second option being an incorrect option; selecting only all the three correct options will result in +4 marks. Selecting only two of the three correct options (e.g. the first and fourth options), without selecting any incorrect option (second option in this case), will result in +2 marks. Selecting only one of the three correct options (either first or third or fourth option), without selecting any incorrect option in this case), will result in +1 marks. Selecting any incorrect option(s) (second option in this case), will result in +1 marks. Selecting any incorrect option(s) (second option in this case), will result in +1 marks. Selecting any incorrect option(s) (second option in this case), with or without selection of any correct option(s) will result in -2 marks.
- 37. F or a non-zero complex number z, let  $\arg(z)$  denote the principal argument with  $-\pi < \arg(z) \le \pi$ . Then, which of the following statement(s) is (are) **FALSE?**

(A) 
$$\arg(-1-i) = \frac{\pi}{4}$$
, where  $i = \sqrt{-1}$ 

- (B) The function  $f: \mathbb{R} \to (-\pi, \pi]$ , defined by  $f(t) = \arg(-1+it)$  for all  $t \in \mathbb{R}$ , is continuous at all points of  $\mathbb{R}$ , where  $i = \sqrt{-1}$
- (C) For any two non-zero complex numbers  $z_1$  and  $z_2$ ,

$$\operatorname{arg}\left(\frac{z_1}{z_2}\right) - \operatorname{arg}(z_1) + \operatorname{arg}(z_2)$$

is an integer multiple of  $2\pi$ 

(D) For any three given distinct complex numbers  $z_1$ ,  $z_2$  and  $z_3$ , the locus of the point z satisfying the condition

$$\arg\left(\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}\right) = \pi$$

lies on a straight line.

Sol. (A), (B), (D)

R

(A) 
$$\arg(-1-i) = \frac{-3\pi}{4}$$
  
 $\therefore$  (A) is false  
(B)  $f(t) = \arg(-1+it) = \begin{bmatrix} \pi - \tan^{-1}(t), & t \ge 0\\ -\pi + \tan^{-1}(t), & t < 0 \end{bmatrix}$   
 $\lim_{t \to 0^{+}} f(t) = -\pi$  and  $\lim_{t \to 0^{+}} f(t) = \pi$   
LHL  $\neq$  RHL  $\Rightarrow$  *f* is discontinuous at  $t = 0$   
 $\therefore$  (B) is false  
(C)  $\arg\left(\frac{z_{1}}{z_{2}}\right) - \arg z_{1} + \arg z_{2}$   
 $= 2n\pi + \arg z_{1} - \arg z_{2} - \arg z_{1} + \arg z_{2} = 2n\pi$ , multiple of  $2\pi$   
 $\therefore$  (C) is true  
(D)  $\arg\left(\frac{(z-z_{1})(z_{2}-z_{3})}{(z-z_{3})(z_{2}-z_{1})}\right) = \pi \Rightarrow \frac{(z-z_{1})(z_{2}-z_{3})}{(z-z_{3})(z_{2}-z_{1})} = k, \quad k \in \mathbb{R}$   
 $\Rightarrow \left(\frac{z-z_{1}}{z-z_{3}}\right) = k\left(\frac{z_{2}-z_{1}}{z_{2}-z_{3}}\right) \Rightarrow z, z_{1}, z_{2}, z_{3}$  are concyclic. i.e. *z* lies on a circle.  
 $\therefore$  (D) is false

- 38. In a triangle *PQR*, let  $\angle =PQR=30^{\circ}$  and the sides *PQ* and QR have lengths  $10\sqrt{3}$  and 10, respectively. Then, which of the following statement(s) is (are) TRUE?
  - (A)  $\angle PQR = 45^{\circ}$
  - (B) The area of the triangle *PQR* is  $25\sqrt{3}$  and  $\angle QPR = 120^{\circ}$
  - (C) The radius of the incircle of the triangle *PQR* is  $10\sqrt{3}$  –15
  - (D) The area of the circumcircle of the triangle *PQR* is 100  $\pi$

(A) 
$$\cos 30^\circ = \frac{PQ^2 + QR^2 - PR^2}{2PQ \cdot QR}$$
  

$$\Rightarrow \frac{\sqrt{3}}{2} = \frac{\left(10\sqrt{3}\right)^2 + 10^2 - PR^2}{2 \times 10\sqrt{3} \times 10} \Rightarrow PR^2 = 100 \text{ or } PR = 10$$

$$\therefore \ \angle P = \angle Q = 30^\circ$$

$$\therefore \text{ (A) is false}$$
(B) Area of  $\Delta PQR = \frac{1}{2}PQ \times QR \times \sin 30^\circ = \frac{1}{2} \times 10\sqrt{3} \times 10 \times \frac{1}{2} = 25\sqrt{3}$ 
Also  $\angle R = 180^\circ - 30^\circ - 30^\circ = 120^\circ$ 

$$\therefore \text{ (B) is true}$$

(C) 
$$r = \frac{\Delta}{s} = \frac{25\sqrt{3}}{\left(\frac{10\sqrt{3}+10+10}{2}\right)} = \frac{25\sqrt{3}}{10+5\sqrt{3}} = \frac{5\sqrt{3}}{2+\sqrt{3}} = 5\sqrt{3}\left(2-\sqrt{3}\right) = 10\sqrt{3}-15$$

### $\therefore$ (C) is true

(D) 
$$R = \frac{abc}{4\Delta} = \frac{10\sqrt{3} \times 10 \times 10}{4 \times 25\sqrt{3}} = 10$$

 $\therefore$  Area of circumcircle =  $\pi R^2 = 100\pi$ 

$$\therefore$$
 (D) is true

39. Let  $P_1: 2x + y - z = 3$  and  $P_2: x + 2y + z = 2$  be two planes. Then, which of the following statement(s) is (are) TRUE?

(A) The line of intersection of  $P_1$  and  $P_2$  has direction ratios 1,2,-1

(B) The line 
$$\frac{3x-4}{9} = \frac{1-3y}{9} = \frac{z}{3}$$
 is perpendicular to the line of intersection of  $P_1$  and  $P_2$ 

- (C) The acute angle between  $P_1$  and  $P_2$  is 60°
- (D) If  $P_3$  is the plane passing through the point (4, 2, -2) and perpendicular to the line of intersection of  $P_1$  and  $P_2$ , then the distance of the point (2, 1, 1) from the plane  $P_3$  is  $\frac{2}{\sqrt{3}}$

### Sol. (C), (D)

(A) Direction ratios of line of intersection of two planes will be given by  $\vec{n}_1 \times \vec{n}_2$ .

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 3\hat{i} - 3\hat{j} + 3\hat{k}$$

 $\therefore$  dr's of line of intersection of  $P_1$  and  $P_2$  are 1, -1, 1

 $\therefore$  (A) is false

(B) Given line can be written as

$$\frac{x-\frac{4}{3}}{3} = \frac{y-\frac{1}{3}}{-3} = \frac{z}{3}$$

Clearly this line is parallel to line of intersection of  $P_1$  and  $P_2$ 

 $\therefore$  (B) is false

(C) If  $\theta$  is the angle between  $P_1$  and  $P_2$  then  $\cos \theta = \left| \frac{2 \times 1 + 1 \times 2 + (-1) \times 1}{\sqrt{6}\sqrt{6}} \right| = \frac{3}{6} = \frac{1}{2}$ 

$$\therefore \theta = 60^{\circ}$$

Hence (C) is true

(D) Equation of plane  $P_3$ :

1(x-4)-1(y-2)+1(z+2)=0 or x-y+z=0

Distance of (2, 1, 1) from 
$$P_3 = \frac{2 - 1 + 1}{\sqrt{1 + 1 + 1}} = \frac{2}{\sqrt{3}}$$

- $\therefore$  (D) is true
- 40. For every twice differentiable function  $f : \mathbb{R} \to [-2, 2]$  with  $(f(0))^2 + (f'(0))^2 = 85$ , which of the following statement(s) is (are) TRUE?
  - (A) There exist  $r, s \in \mathbb{R}$ , where r < s, such that f is one-one on the open interval (r, s)
  - (B) There exists  $x_0 \in (-4, 0)$  such that  $|f'(x_0)| \le 1$
  - (C)  $\lim_{x\to\infty} f(x) = 1$
  - (D) There exists  $\alpha \in (-4, 4)$  such that  $f(\alpha) + f''(\alpha) = 0$  and  $f'(\alpha) \neq 0$

### Sol. (A), (B), (D)

(A) f(x) being twice differentiable, it is continuous but can't be constant throughout the domain.

 $\therefore$  We can find  $x \in (r, s)$  such that f(x) is one one.

Hence (A) is true

- (B) By Lagrange's Mean Value theorem for f(x) in [-4, 0], there exists
  - $x_0 \in (-4, 0)$  such that

$$f'(x_0) = \frac{f(0) - f(-4)}{0 - (-4)} \implies |f'(x_0)| = \left|\frac{f(0) - f(-4)}{4}\right|$$
  

$$\because -2 \le f(x) \le 2$$
  

$$\therefore -4 \le f(0) - f(-4) \le 4$$
  

$$\implies |f'(x_0)| \le 1$$
  

$$\therefore \text{ (B) is true}$$

(C) If we consider  $f(x) = \sin(\sqrt{85}x)$  then f(x) satisfies the given condition  $[f(0)]^2 + [f'(0)]^2 = 1$ 

But  $\lim_{x \to \infty} (\sin \sqrt{85}x)$  does not exist

 $\therefore$  (C) is false

(D) Let us consider  $g(x) = [f(x)]^2 + [f'(x)]^2$ 

$$|f'(x)| \le 1$$
  
Also  $|f(x_1)| \le 2$  as  $f(x) \in [-2, 2]$   
 $\therefore g(x_1) \le 5$ , for same  $x_1 \in (-4, 0)$   
Similarly  $g(x_2) \le 5$ , for same  $x_2 \in (0, 4)$   
Also  $g(0) = 85$   
Hence  $g(x)$  has maxima in  $(x_1, x_2)$  say at  $\alpha$  such that  $g'(\alpha) = 0$  and  $g(\alpha) \ge 85$   
 $g'(\alpha) = 0 \Rightarrow 2f(\alpha)f'(\alpha) + 2f'(\alpha)f''(\alpha) = 0$   
 $\Rightarrow 2f'(\alpha)[f(\alpha) + f''(\alpha)] = 0$   
If  $f'(\alpha) = 0 \Rightarrow g(\alpha) = [f(\alpha)]^2$  and  $[f(\alpha)]^2 \le 4$   
 $\therefore g(\alpha) \ge 85$  (is not possible)  
Hence  $f(\alpha) + f''(\alpha) = 0$  for  $\alpha \in (x_1, x_2) \in (-4, 4)$   
 $\therefore$  (D) is true

41. Let  $f : \mathbb{R} \to \mathbb{R}$  and  $g : \mathbb{R} \to \mathbb{R}$  be two non-constant differentiable functions. If

$$f'(x) = \left(e^{(f(x)-g(x))}\right)g'(x) \text{ for all } x \in \mathbb{R},$$

and f(1) = g(2), then which of the following statement(s) is (are) TRUE?

(A)  $f(2) < 1 - \log_e 2$  (B)  $f(2) > 1 - \log_e 2$  (C)  $g(1) > 1 - \log_e 2$  (D)  $g(1) < 1 - \log_e 2$ 

Sol. (B), (C)

Given 
$$f'(x) = e^{(f(x)-g(x))} \cdot g'(x) \implies e^{-f(x)}f'(x) = e^{-g(x)}g'(x)$$

Integrating both sides, we get

$$-e^{-f(x)} = -e^{-g(x)} + c \implies -e^{-f(x)} + e^{-g(x)} = c \implies -e^{-f(1)} + e^{-g(1)} = -e^{-f(2)} + e^{-g(2)} + e^{$$

But given that f(1) = g(2) = 1

$$\therefore -e^{-1} + e^{-g(1)} = -e^{-f(2)} + e^{-1}$$

$$\Rightarrow e^{-f(2)} + e^{-g(1)} = \frac{2}{e} \Rightarrow e^{-f(2)} < \frac{2}{e} \text{ and } e^{-g(1)} < \frac{2}{e}$$

$$\Rightarrow -f(2) < \ln 2 - 1 \text{ and } -g(1) < \ln 2 - 1$$

$$\Rightarrow f(2) > 1 - \ln 2 \text{ and } g(1) > 1 - \ln 2$$

$$\therefore \text{ (B) and (C) are True}$$

42 Let  $f:[0,\infty) \to \mathbb{R}$  be a continuous function such that

$$f(x) = 1 - 2x + \int_{0}^{x} e^{x-t} f(t) dt$$

for all  $x \in [0,\infty)$ . Then, which of the following statement(s) is (are) TRUE?

(A) The curve y = f(x) passes through the point (1, 2)

(B) The curve y = f(x) passes through the point (2, -1)

(C) The area of the region  $\{(x, y) \in [0, 1] \times \mathbb{R} : f(x) \le y \le \sqrt{1 - x^2}\}$  is  $\frac{\pi - 2}{4}$ 

(D) The area of the region  $\left\{ (x, y) \in [0, 1] \times \mathbb{R} : f(x) \le \sqrt{1 - x^2} \right\}$  is  $\frac{\pi - 1}{4}$ 

### Sol. (B), (C)

$$f(x) = 1 - 2x + \int_0^x e^{x-t} f(t) dt \implies f(x) = 1 - 2x + e^x \int_0^x e^{-t} f(t) dt$$
  
$$\Rightarrow f'(x) = -2 + e^x \int_0^x e^{-t} f(t) dt + e^x \left[ e^{-x} f(x) \right] \implies f'(x) = -2 + \left[ f(x) - 1 + 2x \right] + f(x)$$
  
$$\Rightarrow f'(x) - 2f(x) = 2x - 3$$

Its a linear differential equation

$$IF = e^{\int -2dx} = e^{-2x}$$
  
Solution :  $f(x) \times e^{-2x} = \int e^{-2x} (2x-3) dx$   
 $f(x) \times e^{-2x} = \frac{e^{-2x}}{-2} (2x-3) - \int \frac{e^{-2x}}{-2} \times 2 dx$   
 $e^{-2x} f(x) = \frac{e^{-2x}}{-2} (2x-3) + \frac{e^{-2x}}{-2} + c$   
 $f(x) = -x + \frac{3}{2} + \frac{1}{-2} + ce^{2x}$   
 $f(x) = -x + 1 + ce^{2x}$ 

From definition of function, f(0) = 1

$$\therefore 1 = 1 + c \implies c = 0$$
$$\therefore f(x) = 1 - x$$

Clearly curve y = 1 - x, does not pass through (1, 2) but it passes through (2, -1)

 $\therefore$  (A) is false and (B) is true

Also the area of the region

 $1-x \le y \le \sqrt{1-x^2}$ , is shown in the figure, is given by

= Area of quadrant – Area  $\triangle OAB = \frac{1}{4} \times \pi \times 1^2 - \frac{1}{2} \times 1 \times 1 = \frac{\pi - 2}{4}$ 

 $\therefore$  (C) is true and (D) is false



# **SECTION 2 (Maximum Marks : 24)**

- This section contains EIGHT (08) questions. The answer to each question is a NUMERICAL VALUE.
- For each question, enter the correct numerical value (in decimal notation, truncated/rounded-off to the **second decimal place;** e.g. 6.25, 7.00, -0.33, -.30, 30.27, -127.30) using the mouse and the onscreen virtual numeric keypad in the place designated to enter the answer.
- Answer to each question will be evaluated according to the following marking scheme : *Full Marks* : +3 If ONLY the correct numerical value is entered as answer. *Zero Marks* : 0 In all other cases.

43. The Value of 
$$((\log_2 9)^2)^{\frac{1}{\log_2(\log_2 9)}} \times (\sqrt{7})^{\frac{1}{\log_4 7}}$$
 is\_\_\_\_\_\_

Sol. (8)

$$\left( \left( \log_2 9 \right)^2 \right)^{\frac{1}{\log_2(\log_2 9)}} \times \left( \sqrt{7} \right)^{\frac{1}{\log_4 7}}$$
  
=  $\left( \log_2 9 \right)^{2 \times \log_{(\log_2 9)^2}} \times 7^{\frac{1}{2} \times \log_7 4} = \left( \log_2 9 \right)^{\log_{(\log_2 9)} 4} \times 7^{\log_7 2} = 4 \times 2 = 8$ 

44. The number of 5 digit numbers which are divisible by 4, with digits from the set {1,2, 3,4,5} and the repetition of digits is allowed, is

# Sol. (625)

The last 2 digits, in 5-digit number divisible by 4, can be 12, 24, 32, 44 or 52.

Also each of the first three digits can be any of  $\{1, 2, 3, 4, 5\}$ 

Hence 5 options for each of the first three digits and total 5 options for last 2-digits

 $\therefore$  Required number of 5 digit numbers are =  $5 \times 5 \times 5 \times 5 \times = 625$ 

45. Let *X* be the set consisting of the first 2018 terms of the arithmetic progression 1,6,11,..., and *Y* be the set consisting of the first 2018 terms of the arithmetic progression 9,16,23,.... Then, the number of elements in the set *X* U *Y* is \_\_\_\_\_.

### Sol. (3748)

The given sequences upto 2018 terms are

1, 6, 11, 16, ....., 10086 and 9, 16, 23, ....., 14128

The common terms are

16, 15, 86, ..... upto *n* terms, where  $T_n \le 10086$ 

$$\Rightarrow 16 + (n-1)35 \le 10086 \Rightarrow 35n - 19 \le 10086 = n \le \frac{10105}{35} = 288.7$$

∴ *n* = 288

$$\therefore n(X \cup Y) = n(X) + n(Y) - n(X \cap Y) = 2018 + 2018 - 288 = 3748$$

## 46. The number of real solutions of the equation

$$\sin^{-1}\left(\sum_{i=1}^{\infty} x^{i+1} - x\sum_{i=1}^{\infty} \left(\frac{x}{2}\right)^{i}\right) = \frac{\pi}{2} - \cos^{-1}\left(\sum_{i=1}^{\infty} \left(-\frac{x}{2}\right)^{i} - \sum_{i=1}^{\infty} \left(-x\right)^{i}\right)$$
 lying in the interval  $\left(-\frac{1}{2}, \frac{1}{2}\right)$  is \_\_\_\_\_.

(Here, the inverse trigonometric functions  $\sin^{-1} x$  and  $\cos^{-1} x$  assume values in  $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$  and  $[0, \pi]$ , respectively.)

Sol. (2)

$$\sin^{-1}\left(\frac{x^{2}}{1-x}-x\cdot\frac{x}{2}}{1-\frac{x}{2}}\right) = \sin^{-1}\left(\frac{-\frac{x}{2}}{1+\frac{x}{2}}-\frac{-x}{1+x}\right)$$

$$\Rightarrow \frac{x^{2}}{1-x}-\frac{x^{2}}{2-x} = \frac{-x}{2+x}+\frac{x}{1+x} \Rightarrow \frac{x^{2}}{1-x}-\frac{x}{1+x}+\frac{x}{2+x}-\frac{x^{2}}{2-x} = 0$$

$$\Rightarrow \frac{x(x+x^{2}-1+x)}{1-x^{2}}+\frac{x(2-x-2x-x^{2})}{4-x^{2}} = 0$$

$$\Rightarrow \frac{x(x^{2}+2x-1)}{1-x^{2}}+\frac{x(2-3x-x^{2})}{4-x^{2}} = 0$$

$$\Rightarrow x\left[(x^{2}+2x-1)(4-x^{2})+(1-x^{2})(2-3x-x^{2})\right] = 0$$

$$\Rightarrow x\left[x^{3}+2x^{2}+5x-2\right] = 0$$

$$\Rightarrow x = 0 \text{ or } x^{3}+2x^{2}+5x-2 = 0 = p(x) \text{ (say)}$$
We observe  $p(0) < 0$  and  $p\left(\frac{1}{2}\right) > 0$ 

$$\therefore \text{ One root of } p(x) = 0 \text{ lies in } \left(0, \frac{1}{2}\right).$$
Thus two solutions lie between  $-\frac{1}{2}$  and  $\frac{1}{2}$ 

47. For each positive integer *n*, let

$$y_n = \frac{1}{2} ((n+1)(n+2)....(n+n))^{\frac{1}{n}}.$$

For  $x \in \mathbb{R}$ , let [x] be the greatest integer less than or equal to x. If  $\lim_{n \to \infty} y_n = L$ , then the value of [L] is \_\_\_\_\_.

Sol. (1)

$$y_n = \left(\frac{n+1}{n} \cdot \frac{n+2}{n} \cdot \frac{n+3}{n} \dots \frac{n+n}{n}\right)^{1/n}$$
  

$$\Rightarrow \log y_n = \frac{1}{2} \sum_{r=0}^n \log\left(1 + \frac{r}{n}\right) \Rightarrow \left(\lim_{n \to \infty} y_n\right) = \lim_{n \to \infty} \frac{1}{n} \sum_{r=0}^n \log\left(1 + \frac{r}{n}\right)$$
  

$$\Rightarrow \log L = \int_0^1 \log(1+x) dx = \left[x \log(1+x)\right]_0^1 - \int_0^1 \frac{x}{1+x} dx$$
  

$$= \log 2 - \left[x - \log|1+x|\right]_0^1 = \log 2 - 1 + \log 2 = 2\log 2 - 1 = \log 4 - \log e = \log\left(\frac{4}{e}\right)$$
  

$$\therefore L = \frac{4}{e} \Rightarrow [L] = \left[\frac{4}{e}\right] = 1$$

48. Let  $\vec{a}$  and  $\vec{b}$  be two unit vectors such that  $\vec{a}.\vec{b} = 0$ . For some  $x, y \in \mathbb{R}$ , let  $\vec{c} = x\vec{a} + y\vec{b} + (\vec{a}\times\vec{b})$ . If  $|\vec{c}| = 2$  and the vector  $\vec{c}$  is inclined at the same angle  $\alpha$  to both  $\vec{a}$  and  $\vec{b}$ , then the value of 8 cos<sup>2</sup>  $\alpha$  is \_\_\_\_\_.

Sol. (3)

Given  $|\vec{a}| = |\vec{b}| = 1$ ,  $\vec{a}.\vec{b} = 0$ ,  $|\vec{c}| = 2$ 

 $\vec{c}$  makes angle  $\alpha$  with both  $\vec{a}$  and  $\vec{b}$ 

Also, 
$$\vec{c} = x\vec{a} + y\vec{b} + \vec{a} \times \vec{b}$$
  
 $\vec{c}.\vec{a} = 2\cos\alpha \Rightarrow x = 2\cos\alpha$   
 $\vec{c}.\vec{b} = 2\cos\alpha \Rightarrow y = 2\cos\alpha$   
 $|\vec{c}|^2 = \vec{c}.\vec{c} = |(2\cos\alpha)\vec{a} + (2\cos\alpha)\vec{b} + \vec{a} \times \vec{b}|^2$   
 $\Rightarrow (2)^2 = 4\cos^2\alpha + 4\cos^2\alpha + |\vec{a} + \vec{b}|^2 \Rightarrow 4 = 8\cos^2\alpha + 1 \quad (\because |\vec{a} \times \vec{b}| = 1 \times 1 \times \sin 90^\circ = 1)$   
 $\Rightarrow 8\cos^2\alpha = 3$ 

49. Let *a*, *b*, *c* be three non-zero real numbers such that the equation

 $\sqrt{3} a \cos x + 2b \sin x = c, x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ , has two distinct real roots  $\alpha$  and  $\beta$  with  $\alpha + \beta = \frac{\pi}{3}$ . Then, the value of  $\frac{b}{a}$  is\_\_\_\_\_.

Sol. (0.5)

Given that the equation  $\sqrt{3}a\cos x + 2b\sin x = c$ 

has two roots  $\alpha$  and  $\beta$ , such that  $\alpha + \beta = \frac{\pi}{3}$ 

 $\therefore \sqrt{3}a\cos\alpha + 2b\sin\alpha = c \qquad \dots (1)$ 

and 
$$\sqrt{3}a\cos\beta + 2b\sin\beta = c$$
 ... (2)

subtracting equation (2) from (1) we get

$$\sqrt{3}a(\cos\alpha - \cos\beta) + 2b(\sin\alpha - \sin\beta) = 0$$
  
$$\Rightarrow -\sqrt{3}a2\sin\frac{\alpha + \beta}{2}\sin\frac{\alpha - \beta}{2} + 2b \cdot 2\cos\frac{\alpha + \beta}{2}\sin\frac{\alpha - \beta}{2} = 0$$
  
$$\Rightarrow -2\sqrt{3}a\sin\frac{\pi}{6} + 4b\cos\frac{\pi}{6} = 0 \Rightarrow -2\sqrt{3}a \times \frac{1}{2} + 4b\frac{\sqrt{3}}{2} = 0 \Rightarrow \frac{b}{a} = \frac{1}{2} = 0.5$$

50. A farmer  $F_1$  has a land in the shape of a triangle with vertices at P (0, 0), Q (1, 1) and R (2, 0). From this land, a neighbouring farmer  $F_2$  takes away the region which lies between the side PQ and a curve of the form  $y = x^n (n > 1)$ . If the area of the region taken away by the farmer  $F_2$  is exactly 30% of the area of  $\Delta PQR$ , then the value of n is\_\_\_\_\_.

Shaded area 
$$= \frac{30}{100} \times Ar(\Delta PQR)$$
  
 $\Rightarrow \int_0^1 (x - x^n) dx = \frac{3}{10} \times \frac{1}{2} \times 2 \times 1 \Rightarrow \left(\frac{x^2}{2} - \frac{x^{n+1}}{n+1}\right)_0^1 = \frac{3}{10}$   
 $\Rightarrow \frac{1}{2} - \frac{1}{n+1} = \frac{3}{10} \Rightarrow \frac{1}{n+1} = \frac{1}{2} - \frac{3}{10} = \frac{1}{5} \Rightarrow n = 4$ 

# **SECTION 3 (Maximum Marks : 12)**

- This section contains TWO (02) paragraphs. Based on each paragraph, there are TWO (02) questions.
- Each question has FOUR options. ONLY ONE of these four options corresponds to the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme :

Full Marks	:	+3	If ONLY the correct option is chosen.
Zero Marks	:	0	If none of the option is chosen (i.e. the question is unanswered).
Negative Marks	:	-1	In all other cases.

# PARAGRAPH "X"

Let *S* be the circle in the *xy*-plane defined by the equation  $x^2 + y^2 = 4$ .

### (There are two questions based on PARAGRAPH "X", the question given below is one of them)

51. Let  $E_1E_2$  and  $F_1F_2$  be the chords of *S* passing through the point  $P_0(1, 1)$  and parallel to the *x*-axis and the *y*-axis, respectively. Let  $G_1G_2$  be the chord of *S* passing through  $P_0$  and having slope -1. Let the tangents to *S* at  $E_1$  and  $E_2$  meet at  $E_3$ , the tangents to *S* at  $F_1$  and  $F_2$  meet at  $F_3$ , and the tangents to *S* at  $G_1$  and  $G_2$  meet at  $G_3$ . Then, the points  $E_3$ ,  $F_3$ , and  $G_3$  lie on the curve

(A) 
$$x + y = 4$$
  
(B)  $(x - 4)^2 + (y - 4)^2 = 16$   
(C)  $(x - 4) (y - 4) = 4$   
(D)  $xy = 4$ 

# Sol. (A)

Equation of  $E_1E_2$ : y = 1

Equation of  $F_1F_2 = x = 1$ 

Equation of  $G_1G_2: x + y = 2$ 

By symmetry, tangents at  $E_1$  and  $E_2$  will meet on y-axis and tangents at  $F_1$  and  $F_2$  will meet on x-axis

$$E_1 \equiv (\sqrt{3}, 1) \& F_1 \equiv (1, \sqrt{3})$$

Equation of tangent at  $E_1: \sqrt{3}x + y = 4$ 

Equation of tangent at  $F_1: x + \sqrt{3}y = 4$ 

 $\therefore$  Points  $E_3(0, 4)$  and  $F_3(4, 0)$ 

Tangents at  $G_1$  and  $G_2$  are x = 2 and y = 2 intersecting each other at  $G_3(2, 2)$ .

Clearly  $E_3$ ,  $F_3$  and  $G_3$  lie on the curve x + y = 4

52. Let *P* be a point on the circle *S* with both coordinates being positive. Let the tangent to *S* at *P* intersect the coordinate axes at the points *M* and *N*. Then, the mid-point of the line segment MN must lie on the curve

(A) 
$$(x+y)^2 = 3xy$$
 (B)  $x^{2/3} + y^{2/3} = 2^{4/3}$  (C)  $x^2 + y^2 = 2xy$  (D)  $x^2 + y^2 = x^2 y^2$ 

Sol. (D)

Let point *P* be  $(2\cos\theta, 2\sin\theta)$ 

Tangent at  $P: x \cos \theta + y \sin \theta = 2$ 

$$\therefore M\left(\frac{2}{\cos\theta}, 0\right) \text{ and } N\left(0, \frac{2}{\sin\theta}\right)$$
  
Mid point of  $MN = \left(\frac{1}{\cos\theta}, \frac{1}{\sin\theta}\right)$ 

For locus of mid point (x, y) of MN,

$$x = \frac{1}{\cos \theta}, \ y = \frac{1}{\sin \theta}$$
$$\Rightarrow \frac{1}{x^2} + \frac{1}{y^2} = 1 \Rightarrow x^2 + y^2 = x^2 y^2$$

### PARAGRAPH "A"

There are five students  $S_1, S_2, S_3, S_4$  and  $S_5$  in a music class and for them there are five seats  $R_1, R_2, R_3, R_4$  and  $R_5$  arranged in a row, where initially the seat  $R_i$  is allotted to the student  $S_i$ , i = 1, 2, 3, 4, 5. But, on the examination day, the five students are randomly allotted the five seats.



Paper 1

# (There are two questions based on PARAGRAPH "A", the question given below is one of them)

53. The probability that, on the examination day, the student  $S_1$  gets the previously allotted seat  $R_1$ , and **NONE** of the remaining students gets the seat previously allotted to him/her is

(A) 
$$\frac{3}{40}$$
 (B)  $\frac{1}{8}$  (C)  $\frac{7}{40}$  (D)  $\frac{1}{5}$ 

Sol. (A)

No. of dearrangements for 4 students

$$=4!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}\right)=12-4+1=9$$

Total no. of arrangements of seating of 5 students = 5! = 120

$$\therefore$$
 required probability  $=\frac{9}{120}=\frac{3}{40}$ 

54. For i = 1, 2, 3, 4 let  $T_i$  denote the event that the students  $S_i$  and  $S_{i+1}$  do **NOT** sit adjacent to each other on the day of the examination. Then, the probability of the event  $T_1 \cap T_2 \cap T_3 \cap T_4$  is

(A) 
$$\frac{1}{15}$$
 (B)  $\frac{1}{10}$  (C)  $\frac{7}{60}$  (D)  $\frac{1}{5}$ 

Sol. (C)

Total cases = 5! = 120

Favourable cases :

 $\begin{array}{c}
1, 3, 5, 2, 4\\
1, 4, 2, 5, 3
\end{array}^{2}$   $\begin{array}{c}
2, 4, 1, 3, 5\\
2, 5, 3, 1, 4
\end{array}^{3}$   $\begin{array}{c}
3, 1, 4, 2, 5\\
3, 5, 2, 4, 1\\
3, 1, 5, 2, 4
\end{array}^{3}$   $\begin{array}{c}
4, 2, 5, 1, 3\\
4, 2, 5, 3, 1\\
4, 1, 3, 5, 2
\end{array}^{3}$   $\begin{array}{c}
5, 2, 4, 1, 3\\
5, 3, 1, 4, 2
\end{array}^{2}$   $\begin{array}{c}
5, 2, 4, 1, 3\\
5, 3, 1, 4, 2
\end{array}^{2}$   $\begin{array}{c}
1, 3, 5, 2\\
3, 1, 4, 2
\end{array}^{3}$ 

 $\therefore$  required probability  $=\frac{14}{120}=\frac{7}{60}$